ON A HOMOMORPHISM PROPERTY OF CERTAIN JORDAN ALGEBRAS[1]

BY

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1. **Introduction.** The purpose of this paper is to provide a proof of the following basic result.

THEOREM. *Let $\mathfrak{D}$ be any (nonassociative) algebra with a unity element e over a field $\mathfrak{F}$ of characteristic not two and possessing an involution T over $\mathfrak{F}$. Let $\mathfrak{H}_0$ be the algebra of all three rowed T-hermitian matrices over $\mathfrak{D}$ relative to Jordan multiplication. Then, if $\mathfrak{H}_0$ is a homomorphic image of a special Jordan algebra $\mathfrak{H}$, the algebra $\mathfrak{D}$ is associative.*

This result has a number of important consequences. It may be seen to imply that no simple exceptional finite dimensional Jordan algebra of characteristic not two is a homomorphic image of a special[2] Jordan algebra. But there is an exceptional simple Jordan algebra $\mathfrak{H}$ (of dimension 27) over a field $\mathfrak{F}$ of characteristic not two which is generated by three of its elements. Then $\mathfrak{H}$ is a homomorphic image of the free Jordan algebra $\mathfrak{J}_3$ on three generators. It follows that $\mathfrak{J}_3$ *is not special,* and we also know that $\mathfrak{J}_3$ *is not isomorphic to the free special Jordan algebra* $\mathfrak{J}[x, y, z, 1]$ consisting of all *reversible* polynomials[3] in the free associative algebra $\mathfrak{F}[x, y, z, 1]$ of all polynomials on the three generators x, y, z.

2. **Elementary properties of $\mathfrak{H}_0$ and certain subalgebras.** We begin with a brief description of the algebra $\mathfrak{H}_0$ of our theorem. Let $\mathfrak{D}$ be an algebra with a unity element e over a field $\mathfrak{F}$ of characteristic not two. Suppose that $\mathfrak{D}$ has an involution T over $\mathfrak{F}$; that is, an antiautomorphism

$$x \longrightarrow x^T = \bar{x}$$

Received by the editors July 19, 1958.

(1) The work on this paper was begun at Yale University during the Spring of 1957 and was then sponsored in part by the Office of Ordnance Research, United States Army, under Contract DA-19-059-ORD-2329, and the Office of Scientific Research of the Air Force under Contract AF 18(603)58. The work of L. J. Paige was sponsored in part by NSF Grant G5255, and that of A. A. Albert by NSF Grant G4792. The results of this paper were presented at the International Congress of Mathematicians in Edinburgh on August 19, 1958 under the title *On Jordan algebras with three generators.*

(2) The case where $\mathfrak{H}$ is finite-dimensional was considered in an earlier note entitled *A property of special Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) pp. 624–625.

(3) The identification of the free special Jordan algebra on three generators x, y, z with the Jordan algebra of all reversible polynomials in x, y, z was derived by P. M. Cohn, *On homomorphic images of special Jordan algebras*, Canadian Journal of Mathematics vol. 6 (1954) pp. 253–264.

of period 2 over $\mathfrak{F}$. The set of all self-adjoint elements $x=\bar{x}$ of $\mathfrak{D}$ contains the field $e\mathfrak{F}$.

Now consider the set $\mathfrak{D}_3$ of all three rowed matrices with elements in $\mathfrak{D}$ and its subspace $\mathfrak{H}_0$ of all three rowed T-hermitian matrices. Then $\mathfrak{H}_0$ consists of all matrices

$$h = \begin{pmatrix} \alpha & u & v \\ \bar{u} & \beta & w \\ \bar{v} & \bar{w} & \gamma \end{pmatrix}, \tag{1}$$

where α, β, γ are in the subspace of self-adjoint elements of $\mathfrak{D}$ and u, v, w are in $\mathfrak{D}$. Let hk be the ordinary matrix product in $\mathfrak{D}_3$ and define a product $h\cdot k$ by

$$2h\cdot k = hk + kh. \tag{2}$$

Then $\mathfrak{H}_0$ is an algebra relative to the product $h\cdot k$ and it is known[4] that if $\mathfrak{D}$ is not associative the algebra $\mathfrak{H}_0$ is not a special Jordan algebra. In our case, since $\mathfrak{H}_0$ is to be the homomorphic image of a Jordan algebra $\mathfrak{H}$, $\mathfrak{H}_0$ will be an exceptional Jordan algebra if $\mathfrak{D}$ is not associative.

The restriction of $\mathfrak{H}_0$ to T-hermitian or self-adjoint elements of $\mathfrak{D}_3$ under a standard involution is unnecessary and the present discussion can be modified if $\mathfrak{H}_0$ is defined as the self-adjoint elements of $\mathfrak{D}_3$ under a canonical involution of $\mathfrak{D}_3$. In this case, we obtain as a corollary to our theorem the result noted in the previous paragraph. However, the simplicity afforded by assuming that $\mathfrak{H}_0$ consists of T-hermitian elements of $\mathfrak{D}_3$ will be retained for the remainder of the paper.

We shall be interested in the Jordan subalgebra $\mathfrak{J}_0$ of $\mathfrak{H}_0$ generated by the elements

$$x = \begin{pmatrix} e & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & e & e \\ e & 0 & e \\ e & e & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & u & v \\ \bar{u} & 0 & w \\ \bar{v} & \bar{w} & 0 \end{pmatrix} \tag{3}$$

of $\mathfrak{H}_0$ where u, v and w are arbitrary elements of $\mathfrak{D}$. The unity element of $\mathfrak{J}_0$ and of $\mathfrak{H}_0$ is the identity matrix

$$f = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{4}$$

and f is the sum of the three pairwise orthogonal idempotents

(4) See N. Jacobson, *Structure of alternative and Jordan bimodules*, Osaka Math. J. vol. 6 (1954) pp. 1–71.

$$(5) \qquad e_1 = \frac{x^2+x}{2}, \qquad e_2 = \frac{x^2-x}{2}, \qquad e_3 = f - (e_1+e_2) = f - x^2,$$

represented in the representation (1) by

$$e_1 = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Then $\mathfrak{J}_0$ is the vector space direct sum

$$(6) \qquad \mathfrak{J}_0 = \mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3 + \mathfrak{J}_{12} + \mathfrak{J}_{23} + \mathfrak{J}_{13},$$

where $\mathfrak{J}_i$ is the set of all elements h_i of $\mathfrak{J}_0$ such that

$$(7) \qquad h_i \cdot e_i = h_i \qquad (i = 1, 2, 3)$$

while $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ for $i \neq j$ and is the set of all elements h_{ij} of $\mathfrak{J}_0$ such that

$$(8) \qquad 2e_i \cdot h_{ij} = 2e_j \cdot h_{ij} = h_{ij} \qquad (i \neq j;\ i, j = 1, 2, 3).$$

Since $f = e_i + e_j + e_k$ we see that (8) implies that

$$(9) \qquad e_k \cdot h_{ij} = 0 \qquad (i \neq j;\ k \neq i, j;\ i, j, k = 1, 2, 3).$$

We record our assumption that

$$(10) \qquad e_i \cdot e_i = e_i^2 = e_i, \qquad e_i \cdot e_j = 0 \qquad (i \neq j;\ i, j = 1, 2, 3).$$

We shall now characterize the spaces $\mathfrak{J}_i$ and $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ in terms of the *Jordan triple product*. This product $\{ghk\}$ can be defined in terms of elements g, h, k of $\mathfrak{J}_0$ by

$$(11) \qquad \{ghk\} = (g \cdot h) \cdot k + (h \cdot k) \cdot g - (k \cdot g) \cdot h.$$

Write, for an arbitrary element h of $\mathfrak{J}_0$,

$$(12) \qquad h = h_i + h_j + h_k + h_{ij} + h_{jk} + h_{ik},$$

where

$$(13) \qquad h_i \cdot e_i = h_i, \qquad h_i \cdot e_j = 0 \qquad (i \neq j;\ i, j = 1, 2\ 3)$$

and (7) and (8) both hold. Then, we see that

$$\begin{aligned} \{e_i h e_i\} &= (e_i \cdot h) \cdot e_i + (e_i \cdot h) \cdot e_i - (e_i^2 \cdot h) \\ &= 2h_i + \frac{1}{2}(h_{ij} + h_{ik}) - \left\{ h_i + \frac{1}{2}(h_{ij} + h_{ik}) \right\} = h_i. \end{aligned}$$

and that

$$\begin{aligned} 2\{e_i h e_j\} &= 2\{e_j h e_i\} = (e_j \cdot h) \cdot e_i + 2(e_i h) e_j - (e_i \cdot e_j) \cdot h \\ &= 2(e_i \cdot h) \cdot e_j + 2(e_j \cdot h) \cdot e_i = h_{ij} \end{aligned}$$

for $(i \neq j;\ i, j = 1, 2, 3)$. Thus we have derived the following result.

LEMMA 1. *Let $\mathfrak{J}_i$ be the set of all elements h_i of $\mathfrak{J}_0$ such that $h_i \cdot e_i = h_i$, and $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ be the set of all elements $h_{ij} = h_{ji}$ in $\mathfrak{J}_0$ such that $2e_i \cdot h_{ij} = 2e_j \cdot h_{ij} = h_{ij}$. Then every element h of $\mathfrak{J}_0$ is uniquely expressible in the form $h = h_i + h_j + h_k + h_{ij} + h_{jk} + h_{ik}$, where*

$$h_i = \{e_i h e_i\}, \qquad h_{ij} = 2\{e_i h e_j\} = 2\{e_j h e_i\}. \tag{14}$$

If we apply Lemma 1 to the element h of (1) and let

$$u_{12} = \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad w_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & \bar{w} & 0 \end{pmatrix}, \qquad v_{23} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ \bar{v} & 0 & 0 \end{pmatrix} \tag{15}$$

it is easy to see that

$$h_1 = \alpha e_1, \quad h_2 = \beta e_2, \quad h_3 = \gamma e_3, \quad h_{12} = u_{12}, \quad h_{23} = w_{23}, \quad h_{13} = v_{13}.$$

Moreover, for the Jordan product (2), we complete our product relations (7), (8), (9), and (10) by the consequences

$$2u_{12}w_{23} = (uw)_{13}, \qquad 2u_{12} \cdot v_{13} = (\bar{u}v)_{23}, \qquad 2v_{13} \cdot w_{23} = (v\bar{w})_{12} \tag{16}$$

of (2) and (15), where u, v, w are elements of $\mathfrak{D}$ and the indicated products uw, $\bar{u}v$, $v\bar{w}$ are the products in the algebra $\mathfrak{D}$.

Observe that for the generator y of $\mathfrak{J}_0$, the elements

$$2\{e_i y e_j\} = e_{ij} \qquad (i \neq j;\ i, j = 1, 2, 3), \tag{17}$$

where

$$e_{12} = \begin{pmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix}, \qquad e_{13} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}. \tag{18}$$

Also we see that for the generator z of $\mathfrak{J}_0$,

$$2\{e_1 z e_2\} = u_{12}, \qquad 2\{e_2 z e_3\} = w_{23}, \qquad 2\{e_3 z e_1\} = v_{13}. \tag{19}$$

Now note that (16) and (18) imply

$$2a_{12} \cdot e_{23} = a_{13}, \qquad 2b_{12} \cdot e_{13} = \bar{b}_{23}, \tag{20}$$

so that we have the basic formula

$$8(a_{12} \cdot e_{23}) \cdot (b_{12} \cdot e_{13}) = (ab)_{12}. \tag{21}$$

This formula enables us to identify the set of elements d of $\mathfrak{D}$ occurring in elements h_{12} of $\mathfrak{J}_{12}$ as the smallest self-adjoint subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$ containing e, u, v and w. Hence $\mathfrak{J}_0$ can be considered as an algebra $\mathfrak{H}_0$ with $\mathfrak{D}_0$ as the basic algebra. Consequently the mapping

$$d_{12} \to d \tag{22}$$

for elements of $\mathfrak{J}_{12}$ upon $\mathfrak{D}_0$ can be considered as an algebra isomorphism of $\mathfrak{J}_{12}$ upon $\mathfrak{D}_0$ relative to the multiplication defined by (21).

3. **The Jordan algebra $\mathfrak{H}$ and induced homomorphisms.** Consider a special Jordan algebra $\mathfrak{H}$ over a field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{H}$ is a subspace of an associative algebra $\mathfrak{A}$ over $\mathfrak{F}$. Let HK be the associative product for $\mathfrak{A}$ so that $\mathfrak{A}$ is a Jordan algebra $\mathfrak{A}^{(+)}$ relative to the product operation $H \cdot K$ defined by

$$2H \cdot K = HK + KH \tag{23}$$

and $\mathfrak{H}$ is a subalgebra over $\mathfrak{F}$ of $\mathfrak{A}^{(+)}$. Assume that $\mathfrak{N}$ is an ideal of $\mathfrak{H}$ such that $\mathfrak{H} - \mathfrak{N}$ is isomorphic to $\mathfrak{H}_0$. The elements of $\mathfrak{H} - \mathfrak{N}$ are then the cosets

$$H + \mathfrak{N} \qquad (H \text{ in } \mathfrak{H}), \tag{24}$$

and we can then say that every element h in $\mathfrak{H}_0$ is a coset

$$h = H + \mathfrak{N} = H' \tag{25}$$

defined uniquely for every H of $\mathfrak{H}$. Since the mapping $H \to H'$ is a homomorphism and we have both (23) and (2), the property

$$(H \cdot K)' = H' \cdot K' \tag{26}$$

is reflected as

$$(HK + KH)' = H'K' + K'H'. \tag{27}$$

We now derive the following result.

LEMMA 2. *The algebras $\mathfrak{A}$ and $\mathfrak{H}$ may be selected so that $\mathfrak{A}$ has a unity I, I is in $\mathfrak{H}$, and $I + \mathfrak{N} = I' = f$ is given by* (4).

We adjoin I to $\mathfrak{A}$ and obtain an associative algebra $\mathfrak{A}_1 = \mathfrak{A} + I\mathfrak{F}$ which is the vector space direct sum of $\mathfrak{A}$ and the one-dimensional space $I\mathfrak{F}$. We also adjoin I to $\mathfrak{H}$ to obtain a Jordan algebra $\mathfrak{H}_1 = \mathfrak{H} + I\mathfrak{F}$ which is again a vector space direct sum of $\mathfrak{H}$ and $I\mathfrak{F}$. Evidently $\mathfrak{H}_1$ is special and I is in $\mathfrak{H}_1$. Also $\mathfrak{N}$ is an ideal of $\mathfrak{H}_1$ and $\mathfrak{H}_1 - \mathfrak{N}$ is clearly isomorphic to the direct sum $\mathfrak{H}_0 + I'\mathfrak{F}$. But $\mathfrak{H}_0$ has a unity element $f = F'$, where F is in $\mathfrak{H}$ and $I' = F' + G'$ for an idempotent class G' orthogonal to F'. It follows that G' is actually orthogonal to $\mathfrak{H}_0$, $\mathfrak{H}_1 - \mathfrak{N} = \mathfrak{H}_0 \oplus G'\mathfrak{F}$ and $G'\mathfrak{F}$ is an ideal of $\mathfrak{H}_1 - \mathfrak{N}$. Then the elements of the coset $(I - F) + \mathfrak{N} = G + \mathfrak{N}$ form an ideal $\mathfrak{N}_1$ of $\mathfrak{H}_1$ and $\mathfrak{H}_1 - \mathfrak{N}_1$ is isomorphic to $\mathfrak{H}_0$. Also $I = F + G$, $I' = F' = f$ as desired. This completes the proof.

Henceforth we shall assume that $\mathfrak{A}$ and $\mathfrak{H}$ are selected as in Lemma 2.

Suppose that X, Y, and Z are the elements of $\mathfrak{H}$ whose images are the elements x, y, and z of $\mathfrak{H}_0$, and let $\mathfrak{J}$ be the special Jordan subalgebra of $\mathfrak{H}$ generated by I, X, Y and Z. The homomorphism of $\mathfrak{H}$ upon $\mathfrak{H}_0$ induces a homomorphism of $\mathfrak{J}$ upon $\mathfrak{J}_0$ under which

(28) $$X' = x, \qquad Y' = y, \qquad Z' = z, \qquad I' = f,$$

and there is an ideal $\mathfrak{M}$ of $\mathfrak{J}$ such that $\mathfrak{J}-\mathfrak{M}\cong\mathfrak{J}_0$. The relations (24), (25), (26) and (27) are quite valid for this induced homomorphism if $\mathfrak{H}$ is replaced by $\mathfrak{J}$ and $\mathfrak{N}$ by $\mathfrak{M}$.

We wish to give another description of $\mathfrak{J}$. Let

(29) $$\mathfrak{P} = \mathfrak{F}[X, Y, Z, I]$$

be the subset of $\mathfrak{A}$ of all polynomials

(30) $$\phi = \phi[X, Y, Z]$$

in X, Y, Z including the constant polynomials. Every monomial $\phi=\alpha A_1 \cdots A_n$ in $\mathfrak{P}$ is a product of $\alpha\neq 0$ in $\mathfrak{F}$ and factors $A_t=I, X, Y$ or Z. Define the reverse ϕ^* of ϕ to be $\alpha A_n \cdots A_1$. Every element ϕ of $\mathfrak{P}$ is a sum $\phi=\phi_1+\cdots+\phi_r$ of monomials ϕ_i, and we define its reverse to be $\phi^*=\phi_1^*+\cdots+\phi_r^*$. The mapping

(31) $$\phi \to \phi^*$$

is then an *involution over* $\mathfrak{F}$ of the algebra $\mathfrak{P}$. The set $\mathfrak{R}$ of all *reversible* polynomials $\phi=\phi^*$ of $\mathfrak{P}$ is a Jordan algebra. The elements I, X, Y, Z generate the Jordan subalgebra $\mathfrak{J}$ and it is actually known [3] that $\mathfrak{J}=\mathfrak{R}$.

4. **Proof of the theorem.** We shall write $A\equiv B$ for any two elements A and B of $\mathfrak{J}$ if $A-B$ is in $\mathfrak{M}$. Then $A\equiv 0$ if and only if A is in $\mathfrak{M}$ so that $A'=0$ in $\mathfrak{J}_0$. We now propose to characterize certain subspaces $\mathfrak{R}_i$ and $\mathfrak{R}_{ij}$ of $\mathfrak{J}$. We define $\mathfrak{R}_i$ to be the set of all elements H_i of $\mathfrak{J}$ such that

(32) $$H_i' = h_i \qquad (h_i \text{ in } \mathfrak{J}_i \text{ of } \mathfrak{J}_0),$$

and we define $\mathfrak{R}_{ij}=\mathfrak{R}_{ji}$ for $i\neq j$ to be the set of all elements $H_{ij}=H_{ji}$ of $\mathfrak{J}$ such that

(33) $$H_{ij}' = h_{ij} \qquad (h_{ij} \text{ in } \mathfrak{J}_{ij} \text{ of } \mathfrak{J}_0).$$

However in a special Jordan algebra

$$\begin{aligned} 4\{GHK\} &= (GH+HG)K + K(GH+HG) + (HK+KH)G + G(HK+KH) \\ &\quad - (KG+GK)H - H(GK+KH) = 2(GHK+KHG) \end{aligned}$$

so that we have the formula

(34) $$2\{GHK\} = GHK + KHG.$$

We now use Lemma 1 to obtain the following result as an immediate consequence of (14) and (34).

LEMMA 3. *Let F_1, F_2, F_3 be any element of $\mathfrak{J}$ such that $F_i' = e_i$ for $i=1, 2, 3$. Then every element H of $\mathfrak{J}$ is congruent to an element of the form*

(35) $$H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13},$$

where the elements

$$H_i = F_i H F_i \tag{36}$$

are in $\mathfrak{R}_i$, *and the elements*

$$H_{ij} = F_i H F_j + F_j H F_i \tag{37}$$

are in $\mathfrak{R}_{ij}$.

We use the fact that (6) is a vector space direct sum to obtain the following result.

LEMMA 4. *If* $H \equiv H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13}$, *then* H *is in* $\mathfrak{R}_{ij}$ *if and only if* $H_1 \equiv H_2 \equiv H_3 \equiv H_{ik} \equiv H_{jk} \equiv 0$.

In $\mathfrak{J}_0$, $x^3 - x = 0$ so that $X^3 - X \equiv 0$ and (5) implies that, if

$$E_1 = \frac{X^2 + X}{2}, \qquad E_2 = \frac{X^2 - X}{2}, \qquad E_3 = I - (E_2 + E_3), \qquad F_i = E_i^2, \tag{38}$$

then

$$E_i' = F_i' = e_i \qquad (i = 1, 2, 3). \tag{39}$$

We then use Lemma 3 to obtain the following result.

LEMMA 5. *If* H *is any (reversible) polynomial of* $\mathfrak{J}$ *the elements*

$$H_i = E_i H E_i, \qquad H_i^{(0)} = F_i H F_i, \tag{40}$$

are in $\mathfrak{R}_i$, *and the elements*

$$H_{ij} = E_i H E_j + E_j H E_i, \qquad H_{ij}^{(0)} = F_i H F_j + F_j H F_i \tag{41}$$

are in $\mathfrak{R}_{ij}$. *Every element* H *is congruent to each of the sums* $H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13}$ *and* $H_1^{(0)} + H_2^{(0)} + H_3^{(0)} + H_{12}^{(0)} + H_{23}^{(0)} + H_{13}^{(0)}$, *so that* $H_i \equiv H_i^{(0)}$, $H_{ij} \equiv H_{ij}^{(0)}$.

We now derive the following auxiliary vital characterization.

LEMMA 6. *If* G *is any polynomial of* $\mathfrak{P}$ *the element* $F_i G F_j + F_j G^* F_i$ *is in* $\mathfrak{R}_{ij}$. *Every element of* $\mathfrak{R}_{ij}$ *has a representative of the form* $F_i G F_j + F_j G^* F_i$ *for* G *in* $\mathfrak{P}$.

For if G is in $\mathfrak{P}$ the element $H = E_i G E_j + E_j G^* E_i$ is in $\mathfrak{R}$. By Lemma 5 the element $E_i H E_j + E_j H E_i = E_i^2 G E_j^2 + E_j^2 G^* E_i + E_i E_j (G + G^*) E_i E_j$ is in $\mathfrak{R}_{ij}$. But $E_i^2 = F_i$, $M = E_i E_j$ is in $\mathfrak{M}$, $\{M(G+G^*)M\} = M(G+G^*)M \equiv 0$ and so $E_i H E_j + E_j H E_i \equiv F_i G F_j + F_j G^* F_i$ as desired. Lemma 5 implies that every element of $\mathfrak{R}_{ij}$ is congruent to an element $F_i G F_j + F_j G^* F_i$ for $G = G^*$ in $\mathfrak{R}$ and our proof is complete.

Lemma 6 states that the mapping

$$G \to (F_1 G F_2 + F_2 G^* F_1)' \tag{42}$$

maps the space $\mathfrak{P}$ of all polynomials in I, X, Y, Z onto $\mathfrak{J}_{12}$. We already have a mapping (22) of $\mathfrak{J}_{12}$ to $\mathfrak{D}_0$ and so we have the induced mapping

$$G \to G'' = g, \quad \text{where} \quad g_{12} = (F_1GF_2 + F_2G^*F_1)', \tag{43}$$

of $\mathfrak{P}$ to the subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$.

We use the result of (40) with $H = Y$, $Y' = y$ of (17) to see that

$$E_{ij} = F_iYF_j + F_jYF_i, \qquad E'_{ij} = e_{ij}, \tag{44}$$

and then use (20) to see that

$$2H_{12}\cdot E_{23} = H_{13}, \qquad 2K_{12}\cdot E_{13} = K_{23}, \qquad 8(H_{12}\cdot E_{23})\cdot(K_{12}\cdot E_{13}) = L_{12}, \tag{45}$$

for H_{ij}, K_{ij}, L_{ij} in $\mathfrak{R}_{ij}$. The first expression in (45) yields the computation

$$\begin{aligned}(F_1GF_2 &+ F_2G^*F_1)(F_2YF_3 + F_3YF_2) + (F_2YF_3 + F_3YF_2)(F_1GF_2 + F_2G^*F_1)\\ &= [F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1] + [F_1(GF_2F_3Y)F_2 + F_2(YF_3F_2G^*)F_1]\\ &\quad + [F_2(G^*F_1F_2Y)F_3 + F_3(YF_2F_1G)F_2] + F_2[(G^*F_1F_3Y) + YF_3F_1G)]F_2.\end{aligned}$$

The second bracketed expression is in $\mathfrak{R}_{12}$, the third in $\mathfrak{R}_{23}$ and the last in $\mathfrak{R}_2$ by Lemma 5. Since the first and the sum is in $\mathfrak{R}_{13}$ by (45), we use Lemma 4 to see that

$$2(F_1GF_2 + F_2G^*F_1)\cdot E_{23} \equiv F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1. \tag{46}$$

We similarly compute

$$\begin{aligned}(F_1HF_2 &+ F_2H^*F_1)(F_1YF_3 + F_3YF_1) + (F_1YF_3 + F_3YF_1)(F_1HF_2 + F_2H^*F_1)\\ &= [F_2(H^*F_1^2Y)F_3 + F_3(YF_1^2H)F_2] + [F_2(H^*F_1F_3Y)F_1 + F_1(YF_3F_1H)F_2]\\ &\quad + [F_1(HF_2F_1Y)F_3 + F_3(YF_1F_2H^*)F_1] + F_1[(HF_2F_3Y) + (YF_3F_2H^*)]F_1,\end{aligned}$$

and obtain

$$2(F_1HF_2 + F_2H^*F_1)\cdot E_{13} \equiv F_2(H^*F_1^2Y)F_3 + F_3(YF_1^2H)F_2. \tag{47}$$

We finally compute

$$\begin{aligned}&[F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1][F_3(YF_1^2H)F_2 + F_2(H^*F_1^2Y)F_3]\\ &\qquad + [F_3(YF_1^2H)F_2 + F_2(H^*F_1^2Y)F_3][F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1]\\ &\equiv F_1G(F_2^2YF_3^2YF_1^2)HF_2 + F_2H^*(F_1^2YF_3^2YF_2^2)G^*F_1\end{aligned}$$

since the remaining components are in $\mathfrak{R}_{13}$, $\mathfrak{R}_{23}$ and $\mathfrak{R}_3$ and hence in $\mathfrak{M}$.

We have now shown that, if

$$S = F_2^2YF_3^2YF_1^2, \tag{48}$$

then

$$8[(F_1GF_2 + F_2G^*F_1)\cdot E_{23}]\cdot[(F_1HF_2 + F_2H^*F_1)\ E_{13}] \equiv F_1KF_2 + F_2K^*F_1, \tag{49}$$

where

$$K = [G, H] = GSH. \tag{50}$$

The polynomial space $\mathfrak{P}$ is clearly an algebra $\mathfrak{P}_0$ with respect to the product defined by (50) and it follows from (21) that the mapping $G \to G''$ of (43) is an algebra homomorphism of $\mathfrak{P}_0$ onto the subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$. But

$$[[G, H], A] = (GSH)SA = GS(HSA) = [G, [H, A]],$$

and so $\mathfrak{P}_0$ is associative. Hence $\mathfrak{D}_0$ must be associative and so $u(vw) = (uv)w$. Since u, v and w were any elements of $\mathfrak{D}$ the algebra $\mathfrak{D}$ is associative and the proof of our main theorem is complete.

5. **Consequences.** It is well known([5]) that the only exceptional simple Jordan algebra over an algebraically closed field Ω of characteristic not two is the algebra $\mathfrak{H}_0$ defined by selecting $\mathfrak{D}$ to be the unique eight dimensional split Cayley algebra $\mathfrak{C}$ over Ω. Thus the following corollary is an immediate consequence of our theorem when $\mathfrak{F}$ is algebraically closed.

COROLLARY 1. *Let $\mathfrak{H}$ be a simple exceptional Jordan algebra of finite dimension (necessarily 27) over any field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{H}$ is not a homomorphic image of any special Jordan algebra over $\mathfrak{F}$.*

To complete our proof we let Ω be the algebraic closure of $\mathfrak{F}$ and $\mathfrak{H}_1 = \mathfrak{H}_\Omega$ and $\mathfrak{H}_0$ be defined as above for $\mathfrak{D}$ the split Cayley algebra over Ω. Then $\mathfrak{H}_1$ and $\mathfrak{H}_0$ are isomorphic. If $\mathfrak{H}$ were the homomorphic image of a special Jordan algebra $\mathfrak{J}$ it should be clear that $\mathfrak{H}_\Omega$ is such an image whereas $\mathfrak{H}_0$ is not such an image and our proof is complete.

We also have the following result.

COROLLARY 2. *The free Jordan algebra $\mathfrak{J}_3$ on three generators is not special.*

To derive this result it is only necessary to note that the algebra $\mathfrak{H}_0$ of all three rowed hermitian matrices with elements in a Cayley algebra $\mathfrak{C}$ is generated by three of its matrices. This is clearly a consequence of the fact that the matrices x, y, z of (3) generate $\mathfrak{H}_0$ if we select u, v, w as generators of $\mathfrak{C}$. It is well known([6]) that $\mathfrak{C}$ has three generators and so $\mathfrak{H}_0$ has three generators and is a homomorphic image of the algebra $\mathfrak{J}_3$. Our theorem then implies that $\mathfrak{J}_3$ cannot be special.

Thus we learn also that $\mathfrak{J}_3$ is not isomorphic to the free special Jordan algebra $\mathfrak{J}[x, y, z, 1]$ of all reversible elements in the free associative algebra

([5]) See A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 446–447.

([6]) See, for example, the discussion of composition algebras in the paper of A. A. Albert and N. Jacobson, *On reduced exceptional Jordan algebras*, Ann. of Math. vol. 66 (1957) pp. 400–417.

$\mathfrak{F}[x, y, z, 1]$ of all polynomials on three generators. This is, of course, a quite different situation to that which exists in the case of the free Jordan algebra $\mathfrak{J}_2$ on two generators where $\mathfrak{J}_2$ is isomorphic to $\mathfrak{J}[x, y, 1]$. It would then be of great interest to derive the identities which must exist and which are satisfied by all special Jordan algebras but not by $\mathfrak{J}_3$ or by $\mathfrak{H}_0$.

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